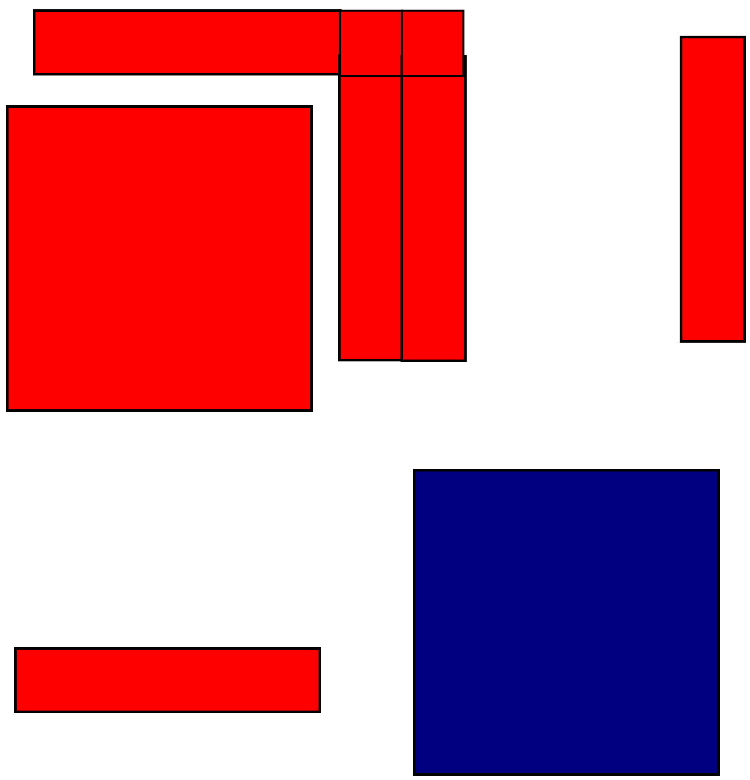
ALGEBRA TILES

TEACHER NOTES AND
STUDENT WORKSHEETS

**FOR A CONCRETE INTRODUCTION
TO THE ABSTRACT CONCEPTS
OF INTEGERS AND ALGEBRA**



**MYRNA INGALLS
UNIONVILLE HIGH SCHOOL
YORK REGION BOARD OF EDUCATION**

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ALGEBRA TILES

Materials Needed For Each Student

- 1 piece of red cardboard, 8 1/2 " x 11"
- 1 piece of blue cardboard, 8 1/2 " x 11"
- scissors
- 1 small plastic ziplock baggie
- 1 copy of each of the student sheets

Instructions to the Teacher

- Provide students with the material to create their own algebra tiles at the time that you review the addition and subtraction of integers. Then you will be able to use the tiles for the units on integers and algebra.
- You may wish to purchase a set of algebra tiles for the overhead projector.
- The concrete → pictorial → symbolic sequence will promote understanding of concepts that often elude students if only the symbolic stage is used in class. The modeling, drawing, and visualization stages provide a change from pencil and paper approaches to mathematics. The **concrete** stage will consist of manipulating cardboard shapes. The **pictorial** stage will be the pencil and paper drawing of the manipulatives. The **symbolic** stage is the written form that uses the symbols of mathematics +, -, x, ÷, x, x^2, x^3, …, 1, 2, 3, …

  At the concrete stage, the results will be described by whole numbers, shape, size, and colour. The narrative style will allow students to work collaboratively. Results will be understood intuitively by most students.

  Although it takes part of a lesson to include the concrete and pictorial stages, the time gained by having students understand the concepts at the intuitive level, saves the time often devoted to drill, review and reteaching. By leading the students, through careful ordering of examples and questions, to the point where they state the patterns being followed in an operation, you will allow students to own the new concepts. They will feel empowered to learn mathematics on their own; their images of themselves as independent learners of mathematics will be positive.

When using a model, it is very important that you be consistent in all of the situations in which you use the model. To help you develop a consistent use of the algebra tiles model, I have included Q and A suggestions. The intention is that you, the teacher, will ask the questions labelled Q orally, and break the question into smaller pieces if no answer is forthcoming from the class.

You may have to refer the students back to the model to get clarification or to have the students add more detail to their first responses. It is important that you structure and time your questions so that the students are the ones who give the answers that have been suggested under A. This process will result in the students' owning of the progress in the class. They will see that the results they are to get in symbolic situations are logical and based on the situations they have modelled. Once you have led the students to verbalize the appropriate pattern, write this pattern down so that students can make a note of the ideas involved.

To help you implement this student-centred approach, I include a series of lessons and worksheets. Each lesson contains a major concept. You may lead the class through one or more of the lessons per day, depending on the length of your classes and on the readiness of your students.

The lesson section is meant to be used during a short teacher-guided introduction. Use overhead manipulatives and use oral questioning at this stage. You may wish to blank out some of the entries in the charts shown with the lessons, then make yourself an overhead copy to use with the class. You could ask students for the appropriate entries and fill those in during the lesson. There is a worksheet that accompanies most lessons. These are designed to provide students with opportunities to practise concepts at the concrete, pictorial, and symbolic levels. They guide students to draw conclusions from the patterns explored. These conclusions can be polished and made into summary notes.

The importance of using the algebra tiles is in giving students a visual, hands-on way of exploring patterns at the introductory stage for a new concept. They give a narrative style for explaining situations and allow students to state the rules of integers and algebra from their own experience. Once students have seen the patterns and stated the rules, the tiles can be put aside. Students can extend rules to more complicated examples using only the symbolic form. It is not intended that complex examples be modelled. By the time you ask students to work with the complex examples, the concept should be understood intuitively because of the way in which you introduced it.

Lesson #1 **Introduction to Integers**

Introduce the notion of integers through the students' real-life experience with situations involving thermometer readings, sea-level, profit and loss, etc.

Introduce the notion of integers through the students' real-life experience with situations involving thermometer readings, sea-level, profit and loss, etc.

Introduce the +4 and -5 notation. Use a variety of types of questions to tie a real-life situation to an integer measure.

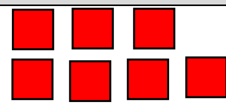
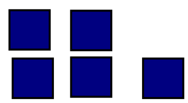
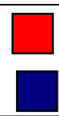
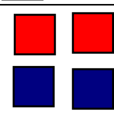
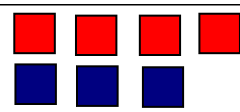
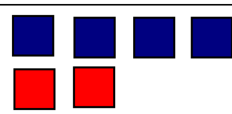
Provide students with the materials needed to create their own set of algebra tiles. Use the "Make your own set of algebra tiles" pattern sheet and instructions.

Lesson #2 **Working with the Tiles**

During this lesson, you introduce students to the narrative style that will allow them to reason through the combinations of integers and terms and to communicate with each other.

Use only the small blue and red squares for this unit on integers.

Work through a series of examples involving the concrete, pictorial, and verbal stages suggested in the chart below.

SHOW	PICTORIAL FORM	RESULT
3 red and 4 red		Red
2 blue and 3 blue		Blue
1 red and 1 blue		Q: Is the result of combining these red or blue? A. Neither
2 red and 2 blue		Neither red nor blue
4 red and 3 blue		The 3 pairs of red and blue give a result that is neither red nor blue. The extra red tile gives a red result
4 blue and 2 red		blue

Once the students are giving consistently correct results by colour, ask:

Q: In some of our results we are getting different amounts of "red" or the various amounts of "blue" from other "red" or "blue" results. How could we distinguish among the various amounts of red or blue possible in a result?

A: We could include numbers in our results.

Q: What are our results in the above questions, using number as well as colour?

A: i) 7 red ii) 5 blue iii) neither iv) neither v) 1 red vi) 2 blue
Continue with examples like those above until all of the students are giving consistently correct results by number and colour.

Organize students in small groups to complete questions like those above. Include the pictorial level as suggested in **Worksheet #1**.

Lesson #3 **Connecting the Tiles to Integers**

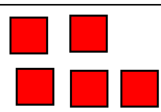
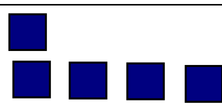
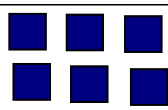
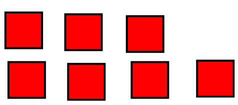
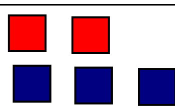
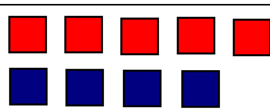
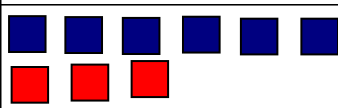
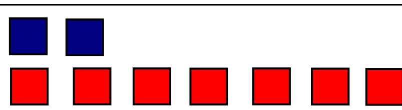
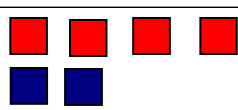
Q: What do the red and blue tiles have to do with integers? We have two types of tiles to represent the integers. How could we divide the integers into two distinct sets? What are the different types of numbers contained within the integers?

A: If we let each red tile be the object we use when we want to investigate the integer positive one, and each blue tile represent the integer negative one, then we will be able to manipulate the tiles to see the patterns involved in combining integers.

Q: How should we interpret a result that is neither red nor blue? What integer is neither positive nor negative?

A: Zero

Using a red tile as +1, a blue tile as -1, and interpreting “neither red nor blue” as 0, investigate the patterns in the following addition of integers questions.

Show this on the overhead	*Ask students which tiles to show.*	*Ask for:*	*Write this beside the question.*
Symbolic form of the question	Pictorial form of the question	Result in number and colour of tiles	Result in symbols
(+2)+(+3)		5 red	+5
(-1)+(-4)		5 blue	-5
(-3)+(-3)		6 blue	-6
(+4)+(+3)		7 red	+7
(+2)+(-3)		1 blue	-1
(+5)+(-4)		1 red	+1
(-6)+(+3)		3 blue	-3
(-2)+(+7)		5 red	+5
(+4)+(-2)		2 red	+2

Once the students are giving consistently correct results, organize students into groups of 2 or 3 to complete **Worksheet #2**.

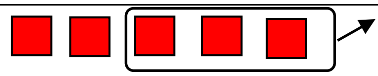
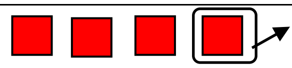
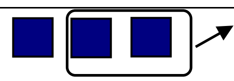
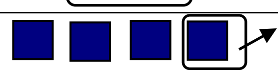
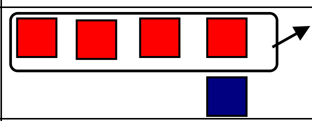
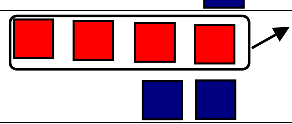
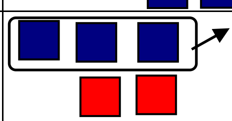
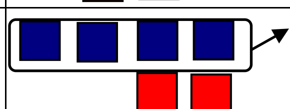
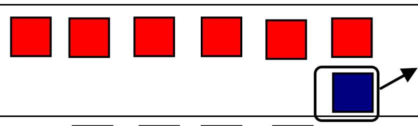
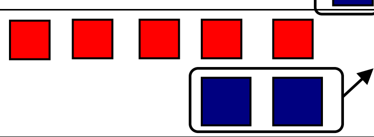
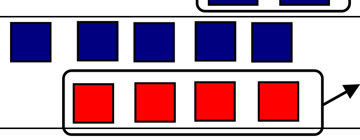
Lesson #4 **Subtraction of Integers**

Q: What is the meaning of “subtraction”?

A: Take away, reduce, remove, the opposite of addition.

Q: How could we use the tiles to model subtraction questions?

A: Start with the model for the first integer and take the model of the second integer away.

Show this on the overhead:	*Ask students which tiles to use*	*Orally ask for:*	*Write this beside the question*
Symbolic form of the question	Pictorial form of the model	Result in number and colour of tiles	Result in symbolic
(+5)-(+3)		2 red	+2
(+4)-(+1)		3 red	+3
(-3)-(-2)		Blue	-1
(-4)-(-1)		3 blue	-3
(+3)-(+4)		1 blue	-1
(+2)-(+4)		2 blue	-2
(-1)-(-3)		2 red	+2
(-2)-(-4)		2 red	+2
(+5)-(-1)		6 red	+6
(+3)-(-2)		5 red	+5
(-1)-(+4)		5 blue	-5

Once students are giving consistently correct responses, organize them into small groups to complete **Worksheet #3**.

Lesson #5 **Mathematical Shorthand**

- In our work to date, we have written positive integers with + signs in front and negative integers with – signs in front. We have also used the + sign to mean the operation of addition and the – sign to mean the operation of subtraction. Now that we have clearly established how we add and subtract positive and negative integers, it is time to look at some mathematical shorthand.
- We noticed in worksheet #3 that we get the same result in (+4)-(+5) as we do in (+4)+(-5). Since this is the case, mathematicians agree that the symbols +4-5 can be read as either "positive four subtract positive five" or as "positive four add negative five".

Also the questions (+5)-(+3) and (+4)+(+1) which have been written in integer form are the same as the questions 5-3 and 4+1 that were understood in elementary school. When an integer is positive, the + sign can be understood rather than written. Therefore, our original question (+4)-(+5) or (+4)+(-5) is normally written 4-5. You are meant to understand that the 4 is positive. Remember that the "-5" part can be read as "subtract positive 5" or as "add negative 5".

- It will depend on the context of a question which way we will choose to read a – sign.

e.g. -2-3 could be expressed as
(i) negative two add negative three, or
(ii) negative two subtract positive three, or
(iii) negative two subtract three, or
(iv) minus two minus three.

Each of these ways of reading the question should bring to you a mental image of the tiles and how to combine them. For most people, the addition model is easier to picture in their minds. For that reason, we will refer to the addition model rather than the subtraction model in future questions. Subtraction is not needed as an operation now that we know how to add a positive or a negative integer to any other.

- Show the following longhand forms on the overhead and ask for the shorthand equivalent form and the result.

Longhand form of a question	Shorthand	Result
(+4)+(-3)	4-3	1
(+5)-(+2)	5-2	3
(-3)+(+7)	-3+7	4
(-2)+(-8)	-2-8	-10
(+10)-(+15)	10-15	-5
(+12)-(+7)	12-7	5
0+(-5)	0-5	-5
(+9)+(+6)	9+6	15
(+8)-0	8-0	8
(+8)+(-5)+(-2)	8-5-2	1
(-10)-(+8)+(+7)	-10-8+7	-11
Elicit responses	10-12	Elicit responses
Elicit responses	-6+20	Elicit responses
Elicit responses	-5-4	Elicit responses
Elicit responses	9-7	Elicit responses

Once students are giving consistently correct responses, have them work on appropriate textbook exercises.

Lesson #6 **Multiplication of Integers**

Recall:

i) In elementary school you saw that the quick way of doing repeated additions was to multiply.

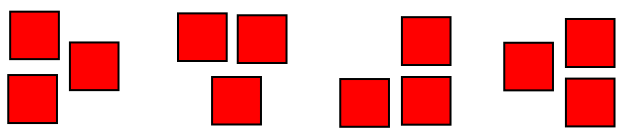
e.g. 5+5+5+5+5+5+5 = 7x5. You start with nothing and add seven groups of five.
3+3+3+3 = 4x3 Start with nothing and add four groups of three.
2+2+2 = 3x2 Three groups of two

ii) The order in which you multiply does not matter.
e.g. 4x3 = 3x4, 5x7 = 7x5. 9x2 = 2x9

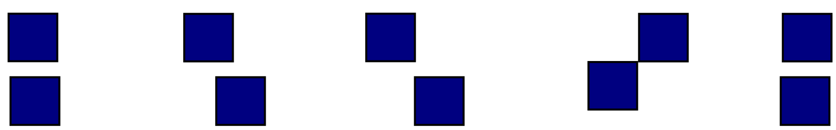
iii) There are three ways of reading a – sign depending n the context. They are: “negative”, “opposite of”, “subtract”.

We will use all of these ideas to investigate the patterns in the multiplication of integers.

4(3) means the same as starting with zero and adding 4 groups of 3. We do not need the zero, but it suggests that, in our concrete model, we are starting with zero and then putting four groups of three red tiles on the table.



Similarly, 5(-2) means start with zero and add five groups of negative two. Our model would have five groups of two blue tiles.



(-2)(5), at first glance, may seem harder to visualize than the example above. However, once we recall the fact that the order of multiplication does not matter, we see that (-2)(5) =(5)(-2)=-10.

Another way of looking at (-2)(5) would be to use the “opposite” interpretation of the – sign. Then we would say that (-2)(5) is “the opposite of two times five”. We know that 2x5=10 and that the opposite of 10 is -10.

A third way of looking at (-2)(5) could include the interpretation “subtract” for the – sign. That is, (-2)(5) is “zero subtract two groups of five”. Recall from the subtraction model that, if we are to remove groups of five from the model, we must have those groups of five present to start with. But, we are to start with zero. How is this possible? We use the familiar zero principle that says: if we have an equal number of red tiles and blue tiles to start, then we are starting with zero. The questions would then look like:

A fourth way of considering (-2)(5) would be to look at the pattern development through comparing one row, question and answer, to the next row in:

You take one more factor of five in each row than you had in the row above.	(3)(5) = 15 (2)(5) = 10 (1)(5) = 5 (0)(5) = 0 (-1)(5) = ? (-2)(5) = ?	You subtract five from the result in one row to get the result for the next row. Therefore, (-1)(5) = -5 and (-2)(5) = -10

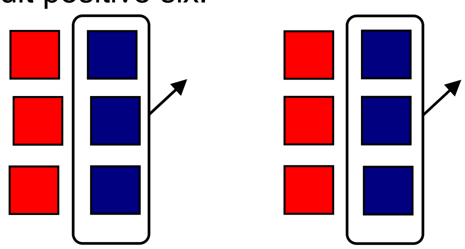
There will be some students who appreciate each of these ways of explaining why (-2)(5)=10.

No matter which of the interpretations we use, we see that we are consistently getting the same result. This should be reassuring to all students!

We can now interpret (-2)(-3) using any of the patterns above:

i) (-2)(-3) could be read "the opposite of two times negative three" to give the opposite of negative six which is positive six.

ii) (-2) (-3) could read "subtract two groups of negative three". This interpretation makes sense only if you think of "zero subtract two groups of negative three". To show this with our model, we would start with six red tiles and six blue tiles (which together have a value of zero), then we would take away two groups of three blue tiles. Can you envision the six red tiles being left as your result? If not, model the scenario with your tiles. We again get the result positive six.



iv) We can get to the question (-2) (-3) by working through the patterns in the questions and answers row to row in:

(3) (-3)	=	-9
(2) (-3)	=	-6
(1) (-3)	=	-3
(-1) (-3)	=	?
(-2) (-3)	=	?

Once students understand the ways of investigating the multiplication of integers, organize them into groups to discover the overall patterns using **Worksheet #4**.

Lesson #7 **Division of Integers**

Recall that division is the inverse operation of multiplication. When you are given a question like 18 ÷ (-3), you could ask what integer multiplies by (-3) to give 18.

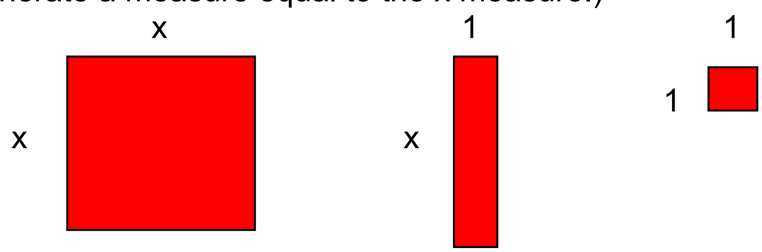
Choose appropriate textbook exercises.

At this point in the course, you will probably work with rational numbers and other topics. You will use the tiles again when you introduce the unit on algebra.

Lesson #8 Modeling Polynomials with Algebra Tiles

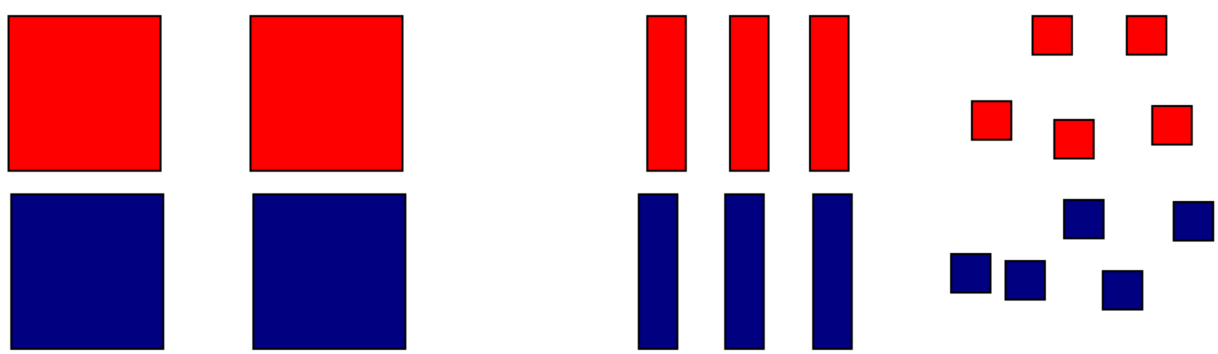
Use all of the shapes you have created for the work on polynomials. The set of tiles in your kit can be used to model polynomials of the form ax^2+bx+c or of the form $ax^2+bxy+cy^2$, where a, b, and c are constants. We recommend that you model only polynomials of the form ax^2+bx+c to establish the appropriate concepts. Students should be able to make the extension to more complex examples at the abstract level, once the concepts are clearly understood.

Show students that each large square has a length of x units and a width of x units, each rectangle has a length of x units and a width of 1 unit, and each small square has a length of 1 unit and a width of 1 unit. (Notice that no integral number of unit lengths will generate a measure equal to the x measure.)



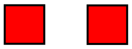
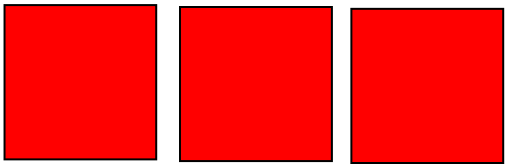
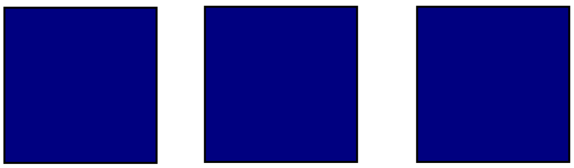
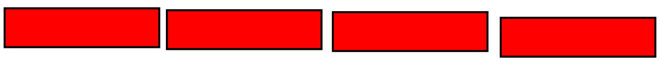
Since our three shapes have different sizes, we should consider what symbolic forms would be consistent with the measurements of the shapes. When we compare two-dimensional shapes, we refer to the area of the shape when we say that one shape is larger than another. The area of each large square is (x)(x) or x^2; the area of each rectangle is (x)(1) or x; the area of each small square is (1)(1) or 1. It would be reasonable, then, to let each large square model a term of size x^2, each rectangle a term of size x, and each small square a term of size 1.

Recall the zero principle to students and extend it to all three shapes. Each of the following combinations of tiles would give a result that is neither red nor blue. Therefore, each of these combinations is a model for zero.



As with the integer tiles, we have both red and blue tiles in each of the three sizes. Recall that if we combine more red than blue tiles of the same size together, our result is red. If we combine more blue than red tiles, our result is blue.

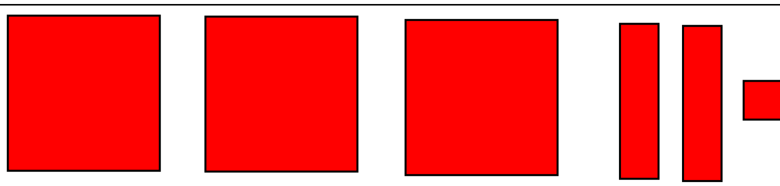
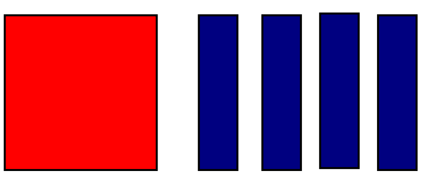
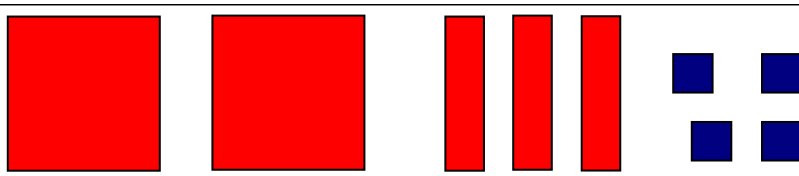
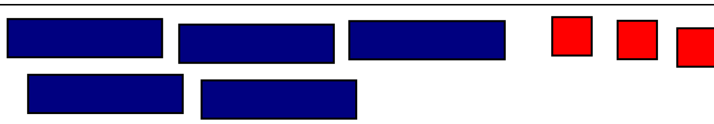
Work through a series of examples involving the concrete, pictorial and verbal stages suggested below.

Show on the overhead	Result by number and colour and size
i) 	2 red small squares
ii) 	3 red large squares
iii) 	3 blue large squares
iv) 	4 red rectangles

Q: How could we record the results symbolically?

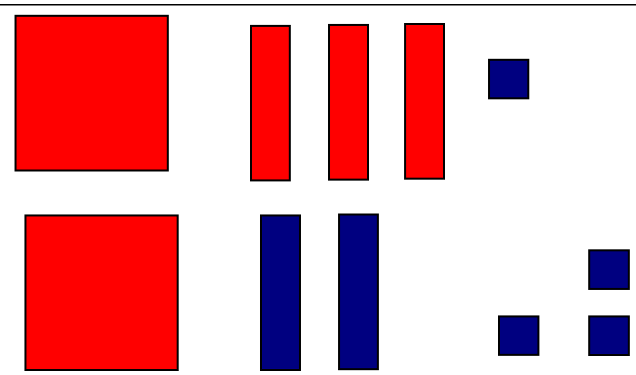
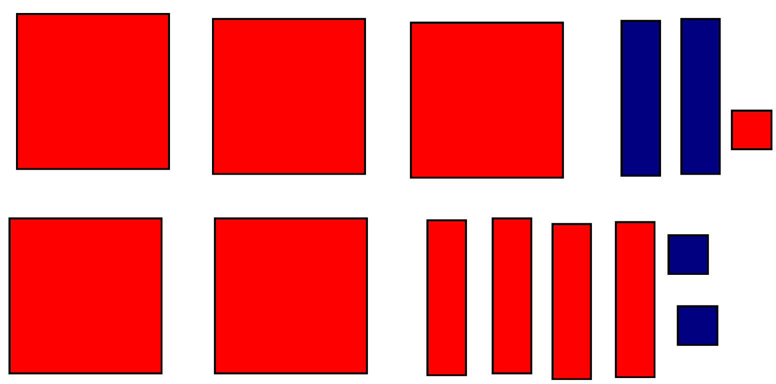
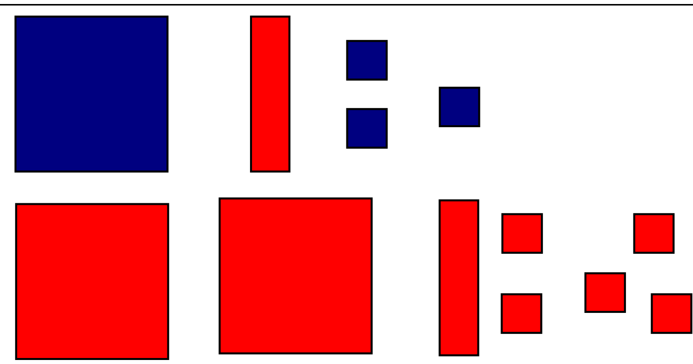
A. i) 2 ii) $3x^2$ iii) $-3x^2$ iv) $4x$

Once students are giving consistently correct responses, advance to examples like the following.

Pictorial form of the model of the polynomial	Symbolic form of the polynomial
	$3x^2+2x+1$
	x^2-4x
	$2x^2+3x-4$
	$-5x+3$

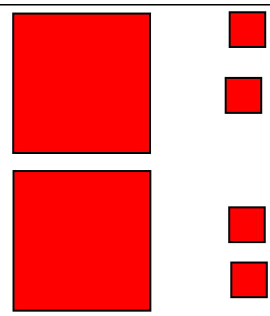
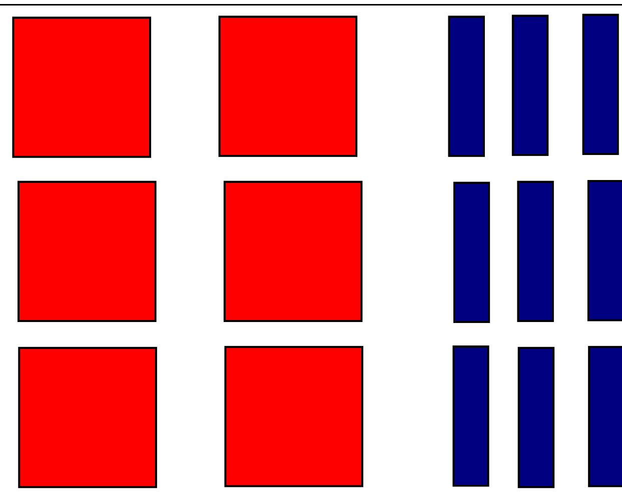
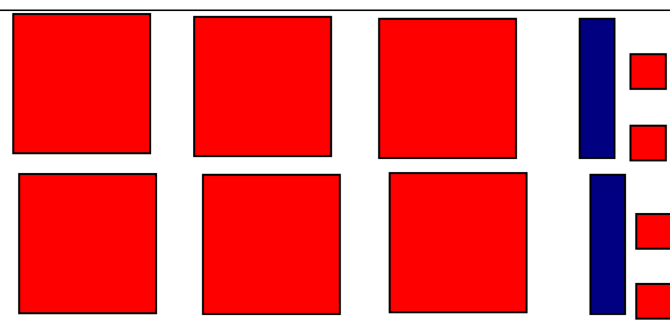
Once students are giving consistently correct responses, organize them into small groups to work on **Worksheet #5**.

Lesson #9 Adding Polynomials using Algebra Tiles

Show this on the overhead	Ask students which tiles to show for:	Ask for:
Symbolic form of two polynomials	Pictorial form so these polynomials	Symbolic form of the result of adding them
$x^2 + 3x - 1$ $x^2 - 2x - 3$		$2x^2+x-4$
$3x^2 - 2x + 1$ $2x^2 + 4x - 2$		$5x^2+2x-1$
$-x^2 + x - 3$ $2x^2 + x + 5$		x^2+2x+2

Once students are giving consistently correct responses, have them complete **Worksheet #6**. After discussing students' results for Worksheet #6, the class could work on textbook exercises.

Lesson #10 **Multiplying a Polynomial by a Monomial (Expanding)**

Using the overhead, ask students how to:			
Model Polynomial	Then	Pictorial Form	Record the results symbolically
i) x^2+2	$2(x^2+2)$		$2x^2+4$
ii) $2x^2-3x$	$3(2x^2-3x)$		$6x^2-9x$
iii) $3x^2-x+2$	$2(3x^2-x+2)$		$6x^2-2x+4$

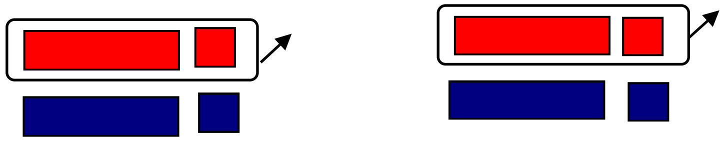
Q: How could we get the results at the symbolic level without using the tiles?

A. You multiply the monomial times each of the terms of the polynomial.

Now tell the class that this process, which removes the brackets, essentially, is called expanding. Ask the students to extend the process to questions like the following.

Expand: 1. $-2(x+1)$

If students ask for the model of this question, remind them of the interpretation $0-2(x+1)$ and show

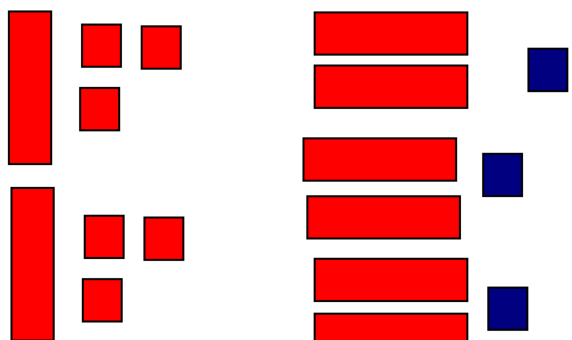
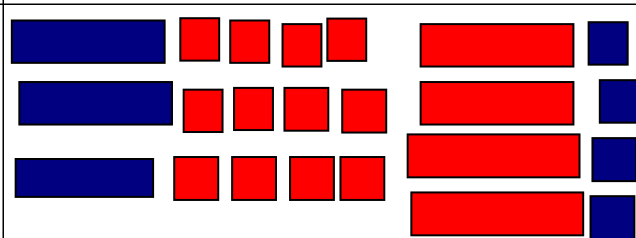


We start with an equal number of red and blue tiles as zero. Then we take away 2 groups of $x+1$. We are left with $-2x-2$.

Encourage students to use just the symbolic level in further questions of this type.

2. $-6(2x+4)$
3. $-5(-3x-2)$

Once students are giving consistently correct results to questions like those above, continue with examples like the following.

Ask students how to model	Pictorial form of the model	Symbolic form of the simplification
$2(x + 3) + 3(2x - 1)$		$8x+3$
$3(-x+4)+4(x-1)$		$x+8$

Q: How could we do these questions symbolically without the tiles?

A: Expand then collect like terms.

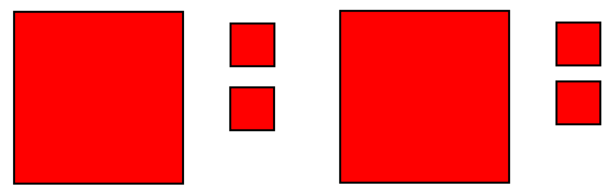
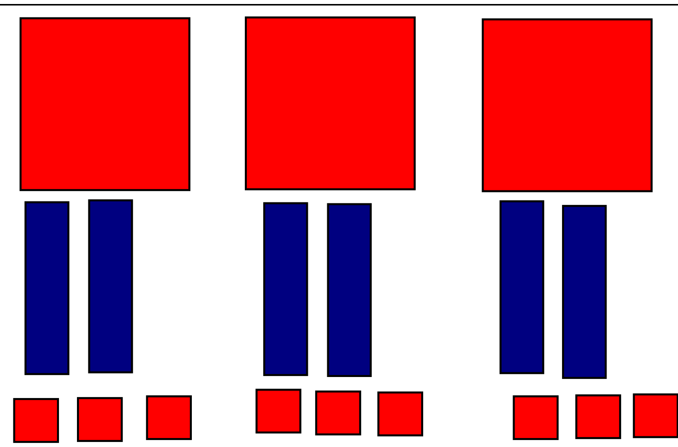
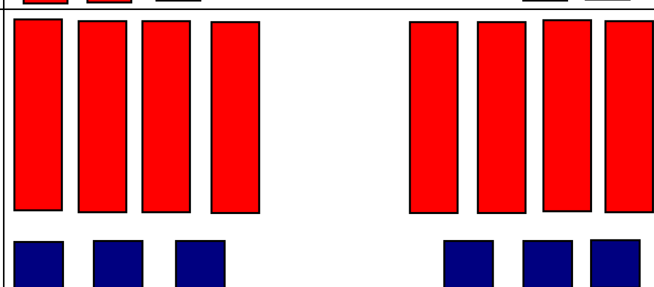
Now simplify

1. $2(x - 3) - 3(4 - x)$ *Remind students that you can read this question as "two times the first binomial add negative three times the second."*
2. $5(a - 5) + (a + 2)$ *Remind students that a coefficient of 1 can be understood.*
3. $(3b^2 - c) - (b^2 + 4c)$

Once students are giving consistently correct responses, assign an appropriate textbook exercise.

Lesson #11 **Division of a Polynomial by a Monomial**

Use the tiles to show this concept visually. Students know intuitively that to divide by two means to split whatever you have into two equal groups, and that to divide by three means to split whatever you have into three equal groups.

Show the model for polynomial	Ask how to model	Pictorial form	Record the results symbolically
$2x^2+4$	$\frac{2x^2+4}{2}$		x^2+2
$3x^2-6x+9$	$\frac{3x^2-6x+9}{3}$		x^2-2x+3
$8x-6$	$\frac{8x-6}{2}$		$4x-3$

Q: How could we do these questions symbolically without the tiles?

A: Each term in the numerator is divided by the monomial in the denominator.

At this stage the students could be assigned a textbook exercise.

<u>Lesson #12</u> <u>Solving Linear Equations in One Unknown</u>

Recall:

(i) an equation is like a balance and that whatever is done to one side of the equation, must be done to the other side. (i.e. whatever one side is added to, subtracted from, multiplied or divided, the same must be done to the other.)

ii) the meaning of "solving" an equation – i.e. Find the value of the variable which will "satisfy" the equation – i.e. When the value of the variable is substituted into all variable positions, both sides of the equation will be equal.

iii) the zero principal

<u>Example 1</u>: Solve the equation 2x + 1 = x + 6.

Show the left side of the equation with tiles.
Show the right side of the equation with tiles.
Create the equation using the tiles.

Q: Assume that we wish the variable to be on the left side. What term is also on that side and how can it be removed?

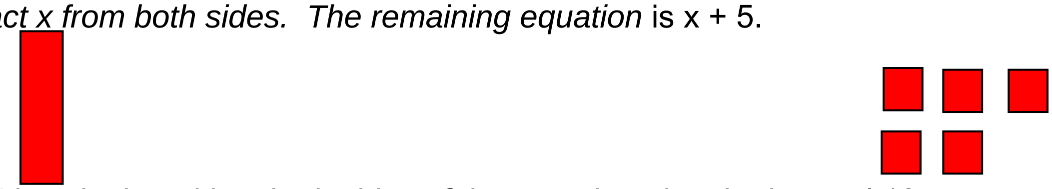
A: The 1 and it can be removed by subtraction, remembering that the same must be done to the right side as well.

Subtract one from both sides. Show that the remaining equation is 2x = x + 5.

Q: To isolate x on one side, what must now be done to both sides?

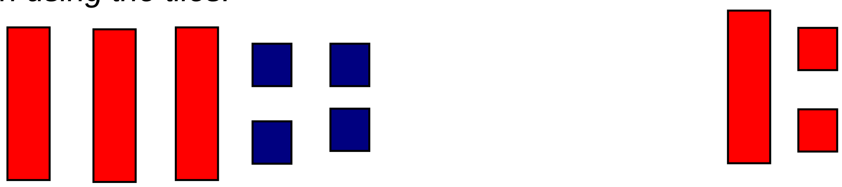
A: Subtract x from both sides.

Subtract x from both sides. The remaining equation is x + 5.



If x + 5 is substituted into both sides of the equation, they both equal 10.
Thus the solution of the equation is that x = 5.

Example 2:   Solve the equation 3x – 4 = x + 2.
Form the equation using the tiles.



Q:	What needs to be done to both sides of the equation?

A:	Add 4 to both sides
Show the result of adding 4 to both sides. 3x-4+4=x+2+4 or 3x = x +6.

Q:	What needs to be done next?

A:	Subtract x from both sides.
Show the result by subtracting x from both sides. 3x-x=x+6-x or 2x = 6.

Q:	How would you find the value of x now?

A:	Divide both sides by 2 and obtain x = 3.

Now we have the solution to equation 3x – 4 = x +2. It is x + 3.

Example 3:   Solve the equation 2x + 5 = 5x – 4.

Q:	What has to be done to both sides of the equation to isolate x?

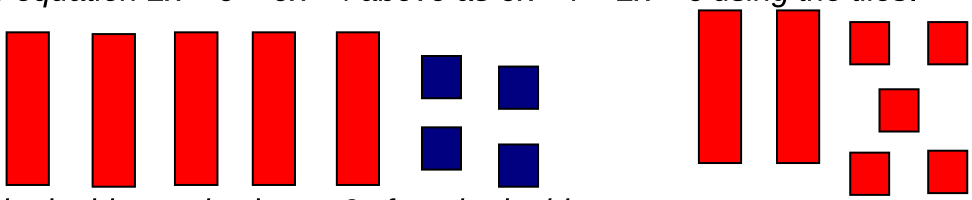
A:	Subtract 5 from both sides and subtract 5x from both sides, or add 4 to both sides and subtract 2x from both sides.

NOTE: Since division by a negative number is difficult to physically illustrate, the teacher should suggest to the students that they reverse the equation to that the larger number of variables is on the left side. With the tiles, this is very easy to show. Another option is to suggest to the students that they work from the side where the larger number of the variables occur. Once the concept of solving equations without using the tiles is developed, the above suggestion is not needed.

Q: Is the equation 2x + 5 = 5x - 4 the same as the equation 5x - 4 = 2x + 5?

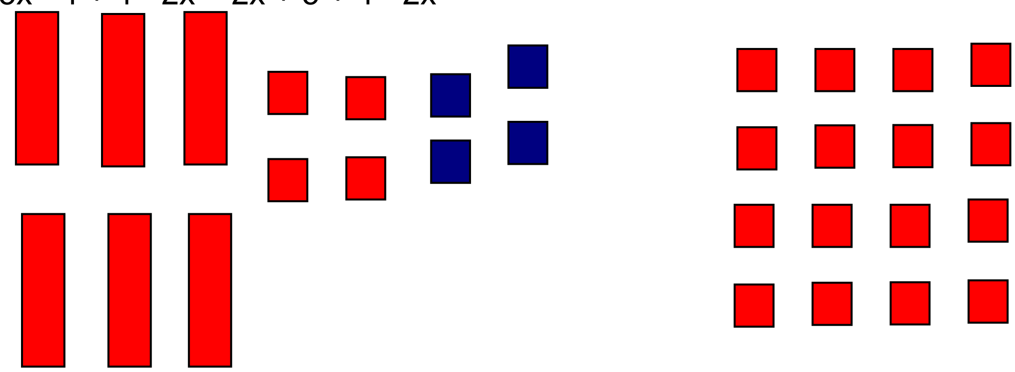
A: Yes, because the equation is like a balance.

Show the equation 2x + 5 = 5x - 4 *above as* 5x - 4 = 2x + 5 *using the tiles*.



Add 4 to both sides and subtract 2x from both sides.

5x - 4 + 4 - 2x = 2x + 5 + 4 - 2x



The result shows 3x = 9.

Q: How do we find the value of x?

A: Divide both sides by 3
(i.e. sort into 3 equal parts)

The result is x = 3.

At this stage the students should be ready to abandon the tiles and work at the symbolic level.

Show this on the overhead	Ask students what to do to both sides	Write this beside the question
Symbolic form of question	What must be done to both sides?	Result in symbols
3x+2+2x+5	Subtract 2 Simplify. Subtract 2x Simplify.	3x+2-2=2x+5-2 3x=2x+3 3 x-2=2x+3-2x x=3
4x-1=2x-3	Add 1 Simplify. Subtract 2x. Simplify. Divide by 2	4x-1+1=2x-3+1 4x=2x-2 4x-2x=2x-2-2x 2x=-2 x=1

Continue with examples until the students are ready to do textbook exercises.

Lesson #13 **Multiplying a Binomial by a Monomial**
(Revisited-in order to introduce lesson #14)

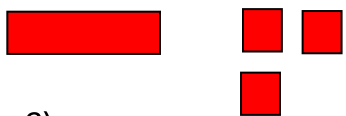
Recall:
- (i) Multiplication of Integers
- (ii) Meanings of Monomials, Binomials, Trinomials, Polynomials

Example 1: Find the value of $2(x + 3)$

Q: What is the meaning of $2(x + 3)$?

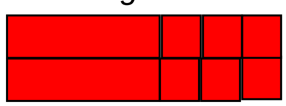
A. The expression $(x+3)$ is to be doubled (multiplied by 2).

Show (x+3) with your tiles.



Show 2(x+3)

Move the tiles together into a rectangle



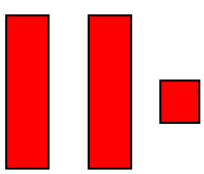
Notice that the rectangle has a width 2 and length $(x+3)$, and that the area of the rectangle is $2x + 6$. Therefore $2(x + 3) = 2x + 6$ could be a model for the width times the length of a rectangle equalling the area.

Example 2: Find the value of $3(2x + 1)$.

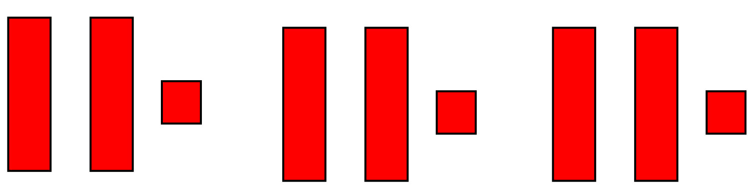
Q: What is the meaning of $3(2x + 1)$.

A: The expression $(2x + 1)$ is to be tripled (multiplied by 3).

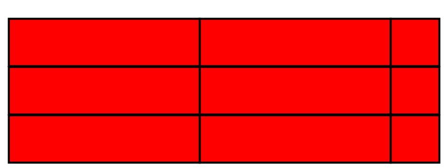
Show $(2x + 1)$



Show $3\ (2x + 1)$



Move the tiles together into a rectangle



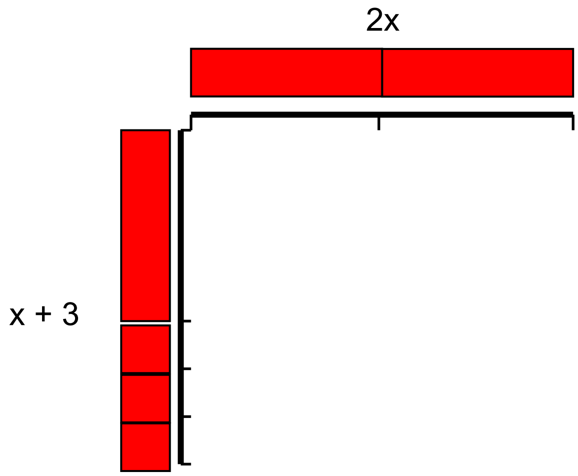
The rectangle formed has the dimensions 3 and $(2x + 1)$ and has an area of $6x + 3$.

Example 3: Find the value of 2x (x + 3).

Q: What would be the dimensions of the rectangle formed if we used tiles?

A: One dimension would be (2x) and the other would be (x + 3).

Show the dimensions.



Complete the rectangle.

Therefore $2x(x + 3) = 2x^2 + 6x$.

Q: What seems to the pattern of multiplying the monomial times the binomial?

A: The term in front, the monomial, is multiplied by each term in the binomial.

Use this idea to expand a variety of questions. Once the students are giving consistently correct results, organize the students into groups of 2 or 3 to complete textbook exercises. Encourage students to refer back to the model if there is a lapse in understanding the concept.

Lesson #14 **Multiplication of Binomials**

Recall: (i) rules of operating with integers.
(ii) multiplying a binomial by a monomial.

Q: How did we use the algebra tiles to illustrate the result of multiplying a binomial by a monomial?

A: By creating a rectangle where the monomial was one dimension and the binomial was the other dimension. The answer was the area of the created rectangle.

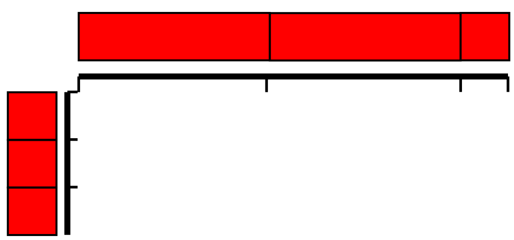
This concept is further developed when a binomial is multiplied by another binomial. Once again we create a rectangle using the two factors (binomials) as the length and width of the rectangle. The expanded form is the area of the rectangle. We determine this area by forming the concrete rectangle out of the tiles. The pieces are chosen so that the resulting rectangle will have the length and width suggested by the factors. Since the area of each tile is known, the area of the rectangle will be the sum of the areas of the individual tiles used.

Let's recall the result of multiplying a binomial by a monomial.

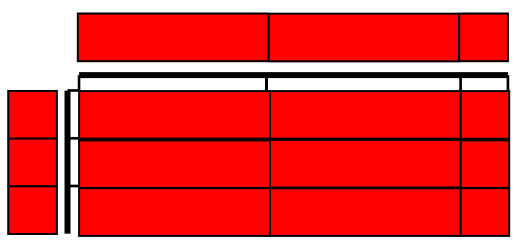
Example: Fine the value of 3(2x + 1).
Set up the dimensions of the rectangle.

2x + 1

3



Complete the rectangle using the tiles.



When completed, this rectangle has width 3 and length 2x+1. Note that its area is 6x+3.
(length) (width) = area
Thus, 3(2x + 1) = 6x + 3 or (2x + 1) (3) = 6x +3.

Procedure:

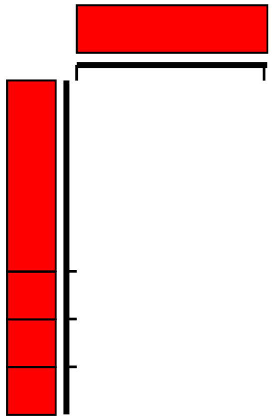
(i) Set up the dimensions of the rectangle by placing the terms of one factor along the top of your workspace and the other factor along the side of the workspace.

(ii) Complete the rectangle (if necessary).

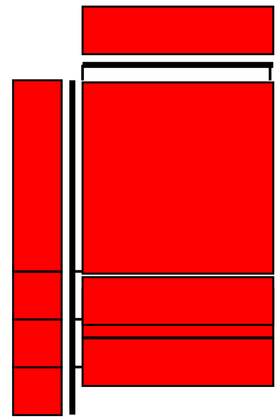
(iii) Read the results by assessing the area sum.

Example 1: Find the value of (x)(x+3)

(i)



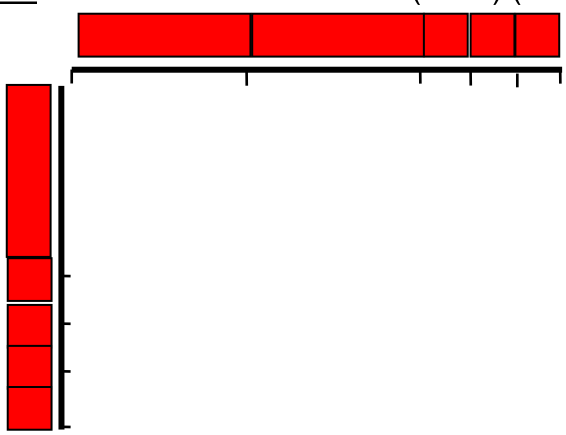
(ii)



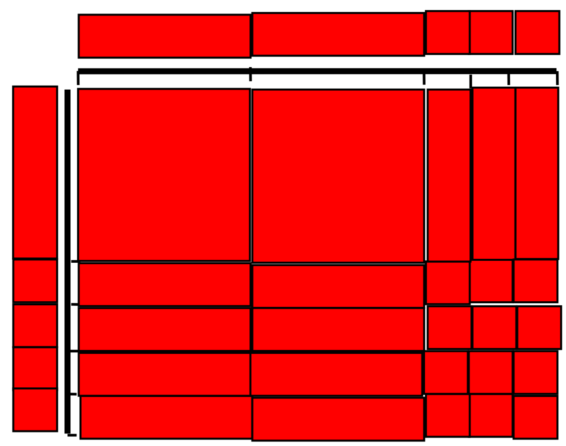
(iii) Therefore $(x)(x + 3) = x^2 + 3x$

Example 2: Find the value of (2x+3) (x+4).

(i)



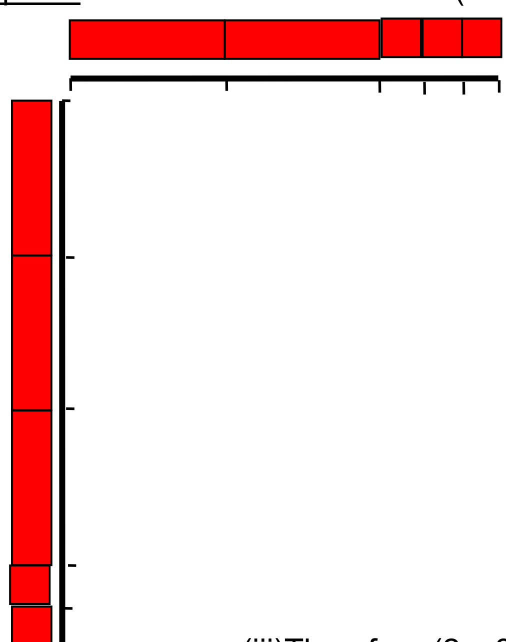
(ii)



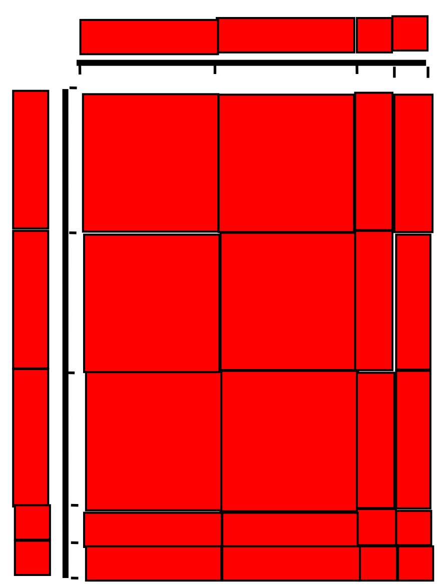
(iii) Therefore $(2x+3)(x+4) = 2x^2 + 11x + 12$

Example 3: Find the value of (3x+2) (2x+3)

(i)



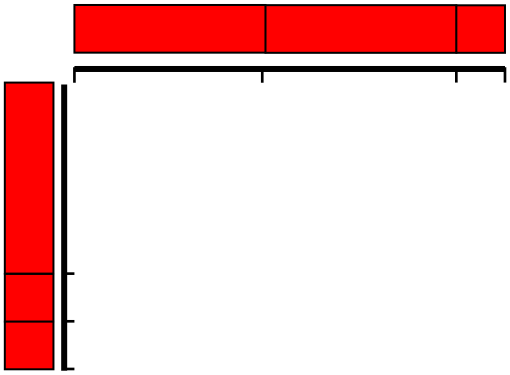
(ii)



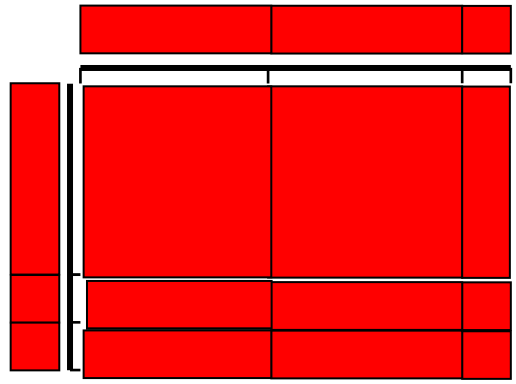
(iii)Therefore $(2x+3)(3x+2) = 6x^2 + 13x + 6$

Example 4: Find the value of $(x+2)(2x+1)$

(i)



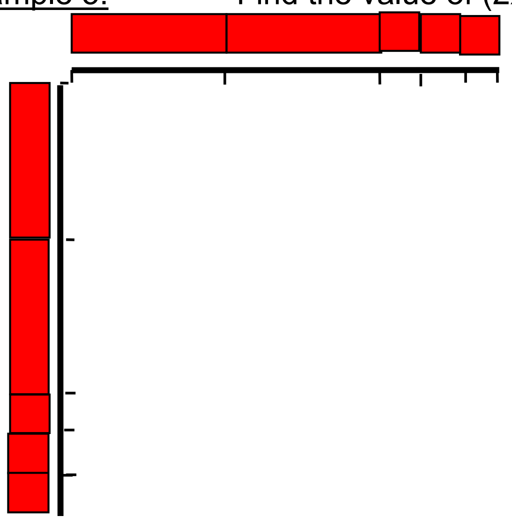
(ii)



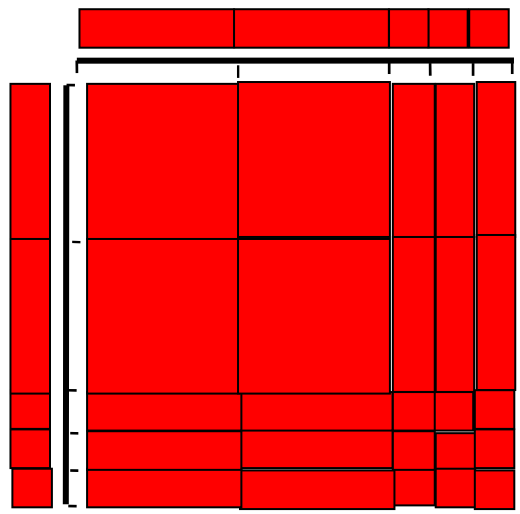
(iii) Therefore $(x+2)(2x+1) = 2x^2 + 5x + 2$

Example 5: Find the value of $(2x+3)^2$

(i)



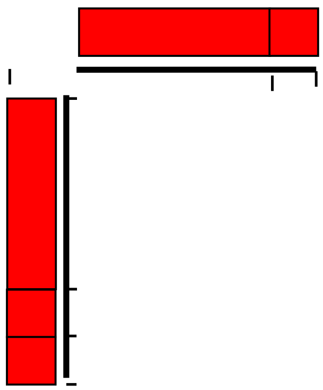
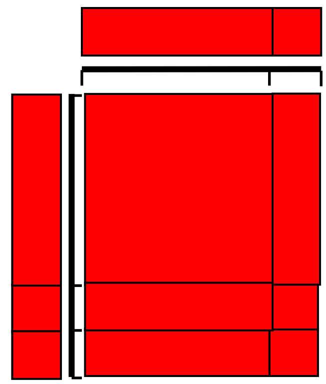
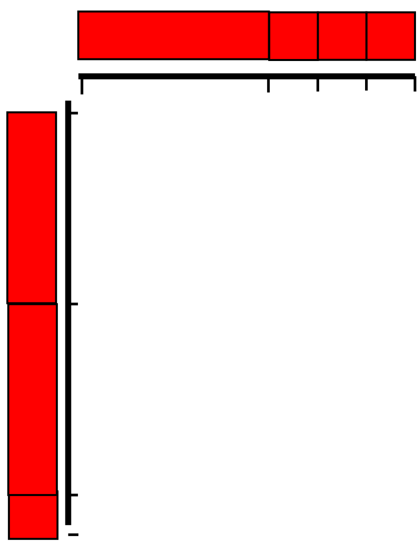
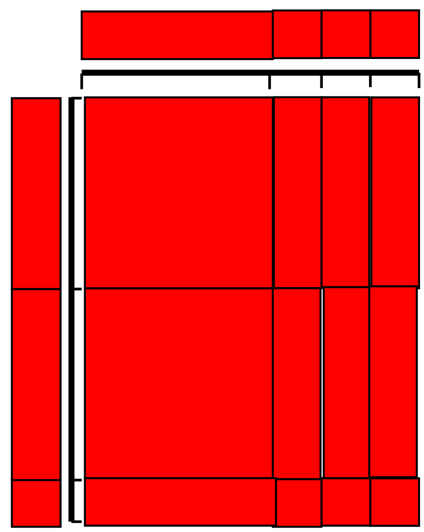
(ii)



(iv) Therefore $(2x+3)^2 = 4x^2 + 12x + 9$

Q: How could the answer be obtained without using the tiles?

A: By multiplying every term in one binomial by every term in the other.

Show this on the overhead	*Ask students which dimensions to use*	*Ask students to complete the rectangle*	*Write this beside the question*
Symbolic form of question	Dimensions of rectangle	Pictorial form in tiles	Result in symbols
(x+2)(x+1)			$x^2 + 3x + 2$
(2x+1) (x+3)			$2x^2 + 7x + 3$

Once the students are giving consistently correct results, organize the students into groups of 2 or 3 to complete **Worksheet #9**

Lesson #15 **Factoring with Algebra Tiles**

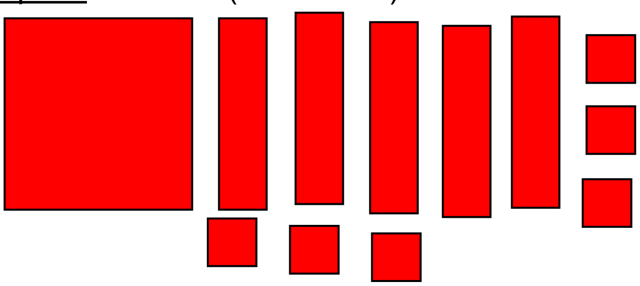
Procedure: (i) select the tiles which represent the product (i.e. area)

(ii) make a rectangular array of the tiles by placing large square tiles in the upper left corner and the unit (1) tiles in the lower right corner

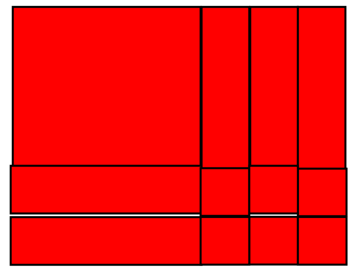
(iii) read the dimensions (i.e. factors) of the completed rectangle

Example 1: Factor $(x^2 + 5x + 6)$.

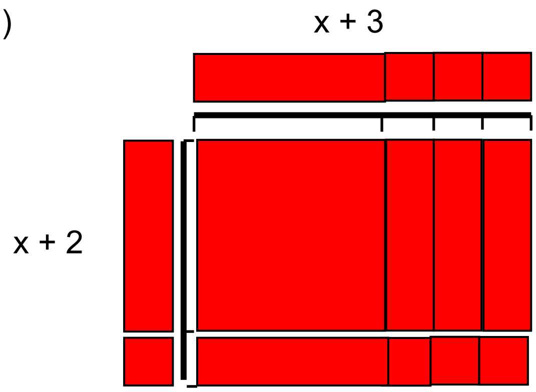
(i)



(ii)



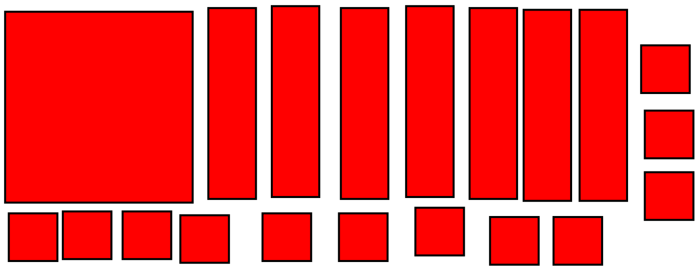
(iii)



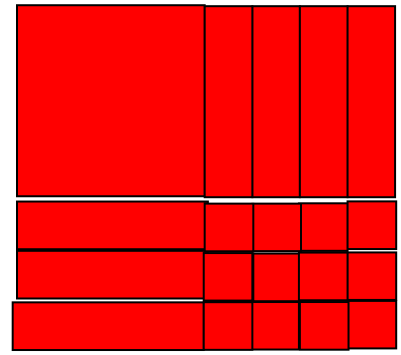
Therefore $(x^2 + 5x + 6)$, factored, equals $(x+2)(x+3)$

Example 2: Factor ($x^2 + 7x + 12$).

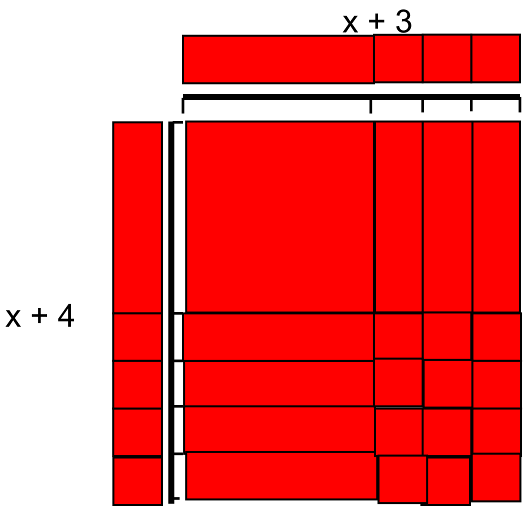
(i)



(ii)



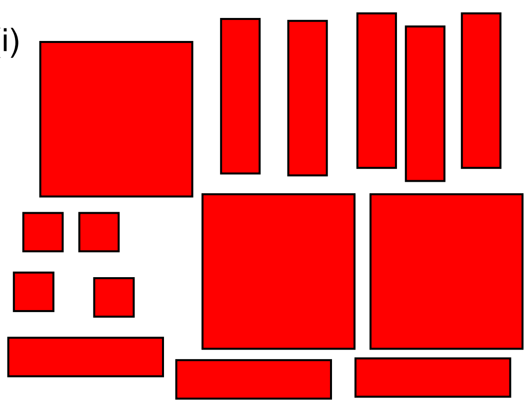
(iii)



Therefore ($x^2 + 7x + 12$), factored, equals $(x+3)(x+4)$.

Example 3: Factor ($3x^2 + 8x + 4$).

(i)

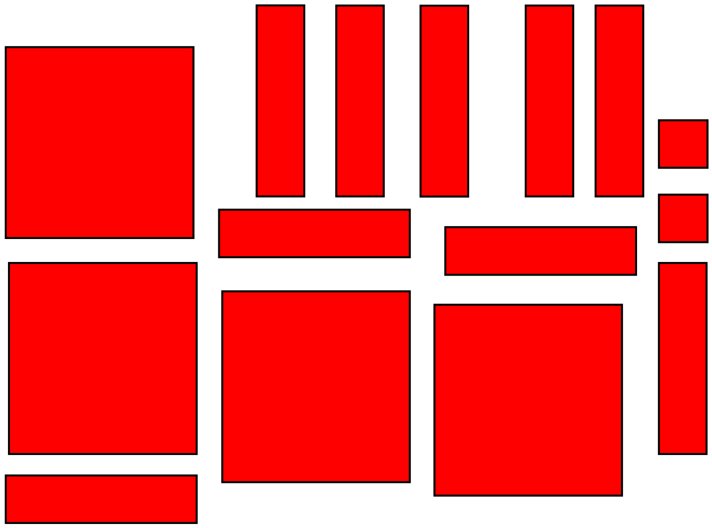


(ii)

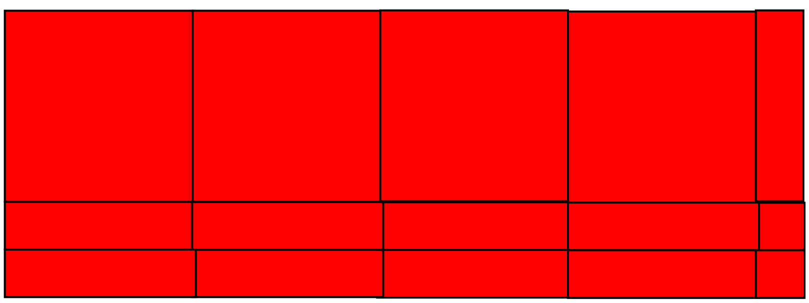
(iii) Therefore ($3x^2 + 8x + 4$), factored, equals $(3x + 2)(x + 2)$.

Example 4: Factor $(4x^2 + 9x + 2)$

(i)



(ii)



(iii) Therefore $(4x^2 + 9x + 2)$, factored, equals $(4x + 1)(x + 2)$.

Once the students have become familiar with using the algebra tiles for factoring, so that they have an appreciation of what factoring means, it is time to explain the patterns of factoring algebraically. This can be done in either of the traditional way of reversing the expansion process, by decomposing the middle term or by using any of the other algebraic methods available.

The algebra tiles are used only for understanding the process of factoring – not for the actual factoring of all trinomials.

Assign textbook exercises for factoring trinomials.

Summary Sheet for Using the Algebra Tiles

1. Integers

- The zero principle: an equal number of positive and negative tiles add to give zero. There are an infinite number of forms of zero
- When adding, put tiles onto the table
- When subtracting, remove tiles from the table
- If the required tiles are not there to be subtracted, add the form of zero that will allow the subtraction you want
- Now that we know how to add both positive and negative integers, we no longer need to use the operation of subtraction
- Interpret multiplication questions as repeated additions, using the words “negative” and “opposite” as needed

2. Adding Polynomials

- Add like terms (tiles) using the rules of integers to get the coefficients. The shape of the tiles determines the type of term –x, x^2, or unit

3. Multiplying Polynomials

- Set up the dimensions of the rectangle, using one factor as the length and the other factor as the width
- Establish the outside tiles
- Complete the rectangle (if necessary)
- Read the area of the rectangle as the result, or product

4. Factoring

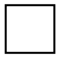
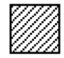
- Select the tiles which represent the given product (i.e. area)
- Make a rectangular array of the of the tiles by placing square tiles in the upper left corner and the unit (1) tiles in the lower right corner
- Read the dimensions (factors) of the completed triangle

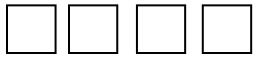
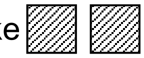
5. Solving Equations in One Variable

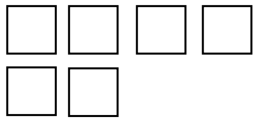
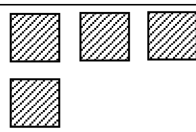
- Model the left and right side of the equation
- Perform the same operation to both sides of the equation
- Work so that the smaller number of variables is removed from both sides of the equation and work to isolate the variable

Worksheet #1

Interpreting Models

Show 1 red tile as  in pictorial form and 1 blue tile as 

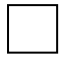
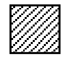
e.g. 4 red would look like  and 2 blue would look like 

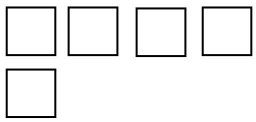
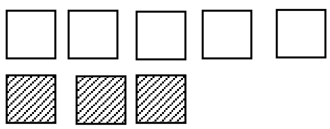
Use your tiles to model	Pictorial form	Result by number and colour
4 red and 2 red		6 red
3 blue and 1 blue		4 blue
3 red and 3 blue		
5 red and 2 blue		
2 red and 3 blue		
4 blue and 3 red		
2 blue and 4 red		
4 red and 4 blue		
		2 blue
		Neither red nor blue

Worksheet #2

Adding integers

Using a red tile as +1, a blue tile as -1, and interpreting "neither red nor blue" as 0, complete the following chart.

Remember that +1 looks like  and -1 looks like  in the pictorial form.

Symbolic form of the question	Pictorial form of the question	Result in number and colour of tiles	Result in symbols
(+4)+(+1)		5 red	+5
(-2)+(-3)			
(+5)+(-3)		2 red	+2
(-2)+(+4)			
(-3)+(+3)			
(-4)+(+1)			
(+5)-(-4)			
(+3)+(+4)			
(+2)-(-2)			
		neither	

Worksheet #2 (cont)

Now answer the following questions by studying your results.

1. What patterns do you see in the questions where you are adding two positive integers or two negative integers?

2. What patterns do you see in the questions where you are adding a positive integer and a negative integer?

3. a) What type of question leads to the answer zero?

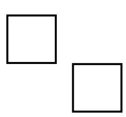
   b) What do you have to add to 3 to generate a result of 0?

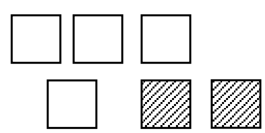
   c) What do you have to add to -7 to generate a result of 0?

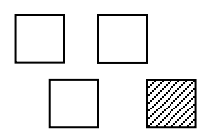
   d) In each of b) and c) we call the number needed the "opposite" of the number given. i.e. Opposites add to give zero. The opposite of 6 is -6, of 7 is -7, of 20 is ____, and the opposite of 99 is ____. In each case you are, symbolically, putting a – sign in front of a number to form its opposite. To be consistent, we should form the opposite of -5 by placing a – sign in front of it. We get - (-5). But we know that the opposite of -5 is _____ since 5 adds to -5 to give 0. Therefore - (-5) = 5. We read "the opposite of negative 5 is five."

   e) What are the two different ways in which we read a – sign in the question above?

4. What integer result do you get from evaluating the following combinations of tiles?

   a) 

   b) 

   c) 

5. Use your results in #4 to show three different pictorial representations of the integer -3.

6. a) What is the minimum number of tiles that could be used to model the integer +1?

   b) Show a model of +1 that uses 5 tiles.

7. a) If you had the model of any integer on the table and added 1 red tile and 1 blue tile to your model, would the resulting integer be the same as the integer you started with or different from the original integer?

   b) If you added two red and two blue tiles to the model of any integer, how would the resulting integer compare to the original integer?

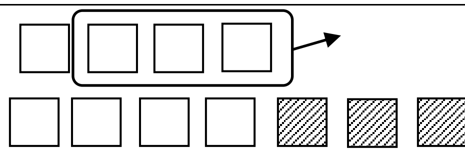
   c) Express the ideas from a) and b) in a sentence.

Worksheet #3

Subtracting Integers

(with comparison to adding integers)

Using a red tile as +1, a blue tile as -1, and a combination of an equal number of red and blue tiles as 0, complete the following chart. To subtract, you take away tiles. To add, you put tiles together.

Symbolic form of the question	Pictorial form of the question	Result in number and colour of tiles	Result in symbols
i) (+4)-(+3) (+4)+(-3)		1 red 1 red	+1 +1
ii) (-3)-(-1) (-3)+(+1)			
(iii) (+2)-(+5) (+2)+(-5)			
(iv) (-5)-(+1) (-5)+(-1)			
v) (+3)-(-2) (+3)+(+2)			

Now answer the following questions:

1. Notice that each subtraction question has an addition question paired with it.

   a) What do you notice about the first integer in each pair of questions?

   b) What do you notice about the results in each pair of questions.

   c) In pair i), what is the effect of subtracting positive three as compared to that of adding negative three?

      ____.

      In pair iii) we see that subtracting +5 is the same as

      ____

   d) What overall pattern can you see from your results in c)?

2. a) If you have the model of any integer and from it take away an equal number of red and blue tiles, how will your result compare to the original integer?

   b) What does this say about subtracting zero from any integer?

3. In our work to date, we have interpreted a – sign in three different ways, depending on the context:

   - In (+4)+(-3) we say "positive four add negative three"
   - In "What is the symbolic way of writing the opposite of 5?", we show -5
   - In 5-2 we say "5 subtract 2".

   a) What would be three different ways of interpreting -4 in a question, depending on the context?

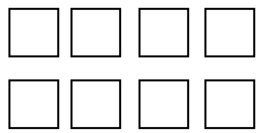
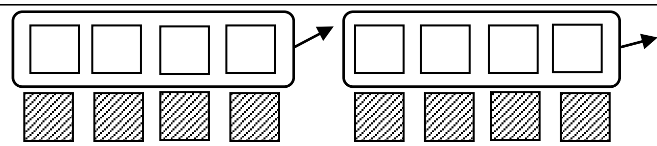
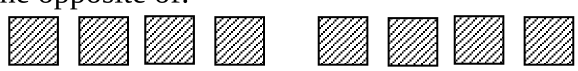
   b) What would be three different ways of interpreting - (-2) in a question, depending on the context?

   c) What other symbols would have the same meaning as -2(-2) in any question?

Worksheet #4

Multiplying Integers

1. Complete the following chart

Symbolic form of multiplication	Pictorial form	Symbolic form of the result
a) (+4)(+2)		8
b) (+2)(-3)		
c) (-1)(+2)		
d) (-2)(-4)	 **Or** the opposite of: 	8
e) (+5)(-1)		
f) (-3)(-2)		

2. Complete the following chart.

Multiplication question	Result	Signs in the question and the result
a) (+2)(+6)	+12	(+)(+)=(+)
b) (+5)(+7)		
c) (+3)(-4)		
d) (+6)(-7)		
e) (-6)(-2)		
f) (-3)(-9)		
g) (-6)(-1)		
h) (-6)(-8)		

3. Use your results in questions 1 and 2 to state a pattern between the signs in a multiplication question and the sign of the answer.

4. Complete the following chart.

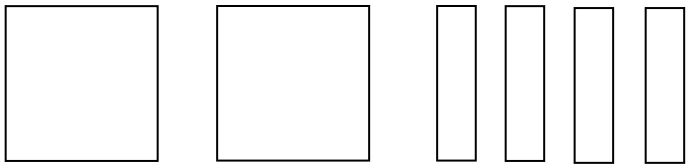
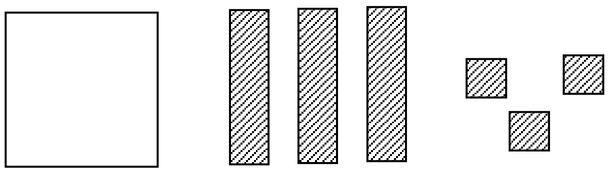
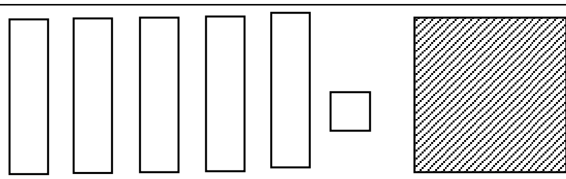
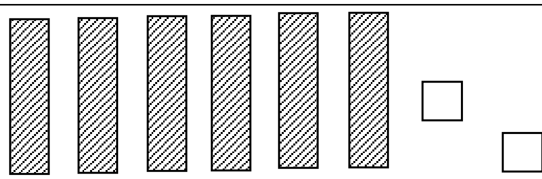
Signs in a multiplication question	Sign of the result
a) (+)(+)(+)	(+)
b) (+)(+)(-)	(-)
c) (+)(-)(-)	
d) (-)(+)(-)	
e) (-)(-)(+)	
f) (-)(-)(-)	
g) (+)(+)(+)(+)	
h) (+)(-)(-)(-)(-)	
i) (-)(-)(-)(-)(-)	
j) 6 negatives	
k) 7 negatives	
l) 8 negatives	
m) an odd number of negatives	
n) an even number of negatives	
o) any number of positives	

5. How could you generalize the pattern for deciding the sign of the result of multiplication integers?

Worksheet #5

Modeling Polynomials with Algebra Tiles

1. Complete the following chart:

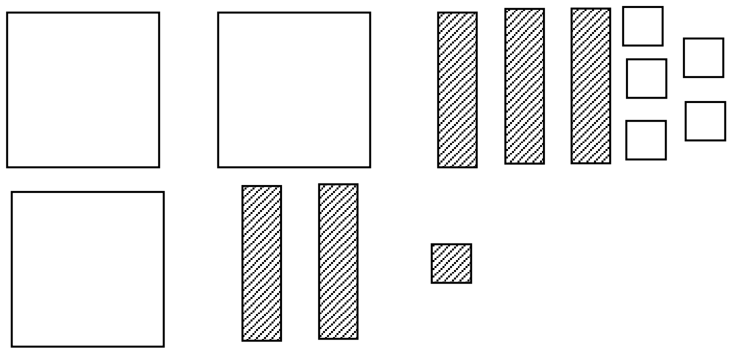
Pictorial form of the polynomial	Symbolic form of the polynomial
a) 	$2x^2+4x$
b) 	
c) 	
d) 	
e)	$5 - 6x$
f)	$x^2 + 3x - 1$
g)	$x - 3x^2$
h)	$-2x^2 + 8$

2. a) What is the minimum number of tiles that could be used to model x^2+3x-1? Show your model.
   b) Add 2 tiles to your model in a) so that you are still modelling x^2+3x-1.
   c) Show 6 different models of x^2+3x-1 that use 9 tiles each

Worksheet #6

Addition of Polynomials

Use your tiles to model the following additions. Then complete the chart.

Symbolic form of the question	Pictorial form of the question	Result in symbolic form
1. $2x^2-3x+5$ x^2-2x-1		$3x^2-5x+4$
2. x^2+4x-6 x^2-2x-1		
3. $5x-3$ $-2x+1$ $x-2$		
4. $7-3x+x^2$ $-4-x-x^2$		

Use your results to answer the following questions.

1. How would we add polynomials together if we were to use only the symbolic form?
2. The tiles that have the same size and share area like; in symbolic form we call them like terms. Reword your observations in #1 using the phrase “like terms”.
3. A term is a mathematical symbol that contains a numerical and/or a literal part. The term $7x^2$ has numerical coefficient 7 and literal coefficient x^2. Term -6abc has numerical coefficient -6 and literal coefficient abc.

Use these examples to help choose the appropriate entries for the following chart.

Term	**Numerical Coefficient**	**Literal Coefficient**
$5x^3$		
$-3y^4$		
$82x^2y^5$		
	-99	p^2

Each term has a degree that depends on the number of literal factors it has. Term $5x^3$; thas degree 3; term $-3y$□ has degree 4; term $82x^2y$ has degree 7. Use these examples to help you complete the following chart.

Term	**Numerical Coefficient**	**Literal Coefficient**	**Degree of the term**
$4a^5$	4	a	5
$-5c^4d^3$			
$24x^8y$			
$8a^2b^4$			
$-6c^5de^3$			

4. Addition questions are often written horizontally, in the symbolic form, rather than vertically.
   e.g. $(5x^2+3x-7)+(3x^2-4x+2)$ means that we are to add the two bracketed polynomials. We will add $3x^2$ to $5x^2$, $-4x$ to $3x$, and 2 to -7. We could communicate this question with fewer symbols if we wrote $5x^2+3X-7+3x^2-4x+2$. In a question written in string form like this, it is easiest to think of collecting like terms if we visually isolate two like terms, remembering to take any sign in front of the term as a positive or negative sign, and tell ourselves to add.
   When we simplify $5x^2+3x-7+3x^2-4x+2$, we get $8x^2-x-5$. Model this question with your tiles to verify the result.

   Simplify the following questions.

   a. $2x^2 + 7x + 8x^2 – x - 6$
   b. $3x^2 - 6x^2 - 4x^2 + 7x^2$
   c. $10ab - 5c + 2ab + 7c$
   d. $9y^4 + 7y^3 – y + 8y - 5y^3 + y^4$
   e. $7pqr + 2r - 3pqr - r$
   f. $8a^2+4a-a^2-9a$

Write, in your own words, the pattern you are using to generate answers in the symbolic form.

Notice that in the questions you have just simplified, you have added both positive and negative terms. Another way of saying what you have done, is that you have added and subtracted terms. There is no need of doing more work on subtraction of terms, since all subtraction questions can more easily be thought of as addition questions.

Make Your Own Set of Algebra Tiles

1. Place the pattern part of this sheet of paper over one piece of red and one piece of blue bristle board.
2. Staple the pattern to the sheets of bristle board at the spots marked.
3. Cut through all three thicknesses.
4. Throw out the pieces of paper pattern and keep all of the blue and red bristle board pieces in a zip lock baggie.